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Differential sandwich results for Wanas operator of analytic functions

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Abstract. In the present article, we determine some subordination and superordination results involving Wanas operator for certain normalized analytic functions defined in the unit disk $\mathbb{U}$. These results are applied to establish sandwich results. Our results extend corresponding previously known results.

1. Introduction

Denote by $H = H(\mathbb{U})$ the collection of analytic functions in the unit disk $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$ and assume that $H[a, n]$ be the subclass of H consisting of functions of the form:

$$f(z) = a + a_n z^n + a_{n+1} z^{n+1} + ... \quad (a \in \mathbb{C},\ n \in \mathbb{N} = \{1, 2, ...\}).$$

Also, let $\mathcal{A}$ be the subclass of H consisting of functions of the form:

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n. \tag{1}$$

Now we recall the principal of subordination between analytic functions, let the functions f and g be analytic in $\mathbb{U}$, we say that the function f is subordinate to g, if there exists a Schwarz function w analytic in $\mathbb{U}$ with $w(0) = 0$ and $|w(z)| < 1$ $(z \in \mathbb{U})$ such that $f(z) = g(w(z))$. This subordination is indicated by $f \prec g$ or $f(z) \prec g(z)$ $(z \in \mathbb{U})$. Furthermore, if the function g is univalent in $\mathbb{U}$, then we have the following equivalent (see [8]), $f(z) \prec g(z) \iff f(0) = g(0)\, and\, f(\mathbb{U}) \subset g(\mathbb{U})$.

Let $\xi, h \in H$ and $\psi(r, s, t; z) : \mathbb{C}^3 \times \mathbb{U} \to \mathbb{C}$. If ξ and

$$\psi\left(\xi(z), z\xi'(z), z^2\xi''(z); z\right)$$

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are univalent functions in $\mathbb{U}$ and if ξ satisfies the second-order differential superordination

$$h(z) \prec \psi\left(\xi(z), z\xi'(z), z^2\xi''(z); z\right), \tag{2}$$

then ξ is called a solution of the differential superordination (2). (If f is subordinate to g, then g is superordinate to f). An analytic function q is called a subordinat of (2), if $q \prec \xi$ for all ξ satisfying (2). An univalent subordinant $\widetilde{q}$ that satisfies $q \prec \widetilde{q}$ for all the subordinates q of (2) is called the best subordinant.

For $\alpha \in \mathbb{R}$, $\beta \geq 0$ with $\alpha + \beta > 0$, $m, \delta \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$ and $f \in \mathcal{A}$, the Wanas operator $W_{\alpha,\beta}^{k,\delta} : \mathcal{A} \to \mathcal{A}$ (see [24]) is defined by

$$W_{\alpha,\beta}^{k,\delta} f(z) = z + \sum_{n=2}^{\infty} \left[\sum_{m=1}^{k} \binom{k}{m} (-1)^{m+1} \left(\frac{\alpha^m + n\beta^m}{\alpha^m + \beta^m} \right) \right]^{\delta} a_n z^n. \tag{3}$$

Remark 1. It should be remarked that the operator $W_{\alpha,\beta}^{k,\delta}$ generalizes some known operators considered earlier:

(1) For $k = 1$, the operator $W_{\alpha,\beta}^{1,\delta} \equiv I_{\alpha,\beta}^{\delta}$ was introduced and studied by Swamy [22],

(2) For $k = \beta = 1$, $\delta = -\mu$, $Re(\mu) > 1$ and $\alpha \in \mathbb{C} \setminus \mathbb{Z}_0^-$, the operator $W_{\alpha,1}^{1,-\mu} \equiv J_{\mu,\alpha}$ was investigated by Srivastava and Attiya [16]. The operator $J_{\mu,\alpha}$ is now popularly known in the literature as the Srivastava-Attiya operator. Various applications of the Srivastava-Attiya operator are found in [15, 17, 18, 19, 20] and in the references cited in each of these earlier works,

(3) For $k = \beta = 1$ and $\alpha > -1$, the operator $W_{\alpha,1}^{1,\delta} \equiv I_{\alpha}^{\delta}$ was investigated by Cho and Srivastava [6],

(4) For $k = \alpha = \beta = 1$, the operator $W_{1,1}^{1,\delta} \equiv I^{\delta}$ was considered by Uralegaddi and Somanatha [23],

(5) For $k = \alpha = \beta = 1$, $\delta = -\sigma$ and $\sigma > 0$, the operator $W_{1,1}^{1,-\sigma} \equiv I^{\sigma}$ was introduced by Jung et al. [7]. The operator I^{σ} is the Jung-Kim-Srivastava integral operator,

(6) For $k = \beta = 1$, $\delta = -1$ and $\alpha > -1$, the operator $W_{\alpha,1}^{1,-1} \equiv L_{\alpha}$ was studied by Bernardi [4],

(7) For $\alpha = 0$, $k = \beta = 1$ and $\delta = -1$, the operator $W_{0,1}^{1,-1} \equiv u$ was investigated by Alexander [1],

(8) For $k = 1$, $\alpha = 1 - \beta$ and $\beta \geq 0$, the operator $W_{1-\beta,\beta}^{1,\delta} \equiv D_{\beta}^{\delta}$ was given by Al-Oboudi [2],

(9) For $k = 1$, $\alpha = 0$ and $\beta = 1$, the operator $W_{0,1}^{1,\delta} \equiv S^{\delta}$ was considered by Sălăgean [13].

It is readily verified from (3) that

$$z\left(W_{\alpha,\beta}^{k,\delta}f(z)\right)' = \left[\sum_{m=1}^{k}\binom{k}{m}(-1)^{m+1}\left(\left(\frac{\alpha}{\beta}\right)^{m}+1\right)\right]W_{\alpha,\beta}^{k,\delta+1}f(z) - \left[\sum_{m=1}^{k}\binom{k}{m}(-1)^{m+1}\left(\frac{\alpha}{\beta}\right)^{m}\right]W_{\alpha,\beta}^{k,\delta}f(z). \tag{4}$$

Very recently, Rahrovi [12], Attiya and Yassen [3], Seoudy [14], Wanas and Majeed [25] and Srivastava and Wanas [21] have obtained sandwich results for certain classes of analytic functions. Motivated by aforementioned works to investigate sufficient condition for f based on Wanas differential operator we define a new subclasses of normalized analytic functions satisfying the following:

$$q_1(z) \prec \left(\frac{W_{\alpha,\beta}^{k,\delta}f(z)}{z}\right)^{\gamma} \prec q_2(z)$$

and

$$q_1(z) \prec \left(\frac{W_{\alpha,\beta}^{k,\delta+1}f(z)}{W_{\alpha,\beta}^{k,\delta}f(z)}\right)^{\gamma} \prec q_2(z),$$

where q_1 and q_2 are given univalent functions in $\mathbb{U}$ with $q_1(0) = q_2(0) = 1$. To establish our main results, we need the following definition and lemmas.

Definition 1 ([8])**.** Denote by Q the set of all functions f that are analytic and injective on $\overline{\mathbb{U}}\backslash E(f)$, where

$$E(f) = \left\{\zeta \in \partial\mathbb{U} : \lim_{z\to\zeta} f(z) = \infty\right\}$$

and are such that $f'(\zeta) \neq 0$ for $\zeta \in \partial\mathbb{U}\backslash E(f)$.

Lemma 1 ([8])**.** *Let q be univalent in the unit disk $\mathbb{U}$ and let θ and ϕ be analytic in a domain D containing $q(\mathbb{U})$ with $\phi(w) \neq 0$ when $w \in q(\mathbb{U})$. Set $Q(z) = zq'(z)\phi(q(z))$ and $h(z) = \theta(q(z)) + Q(z)$. Suppose that*

(1) *$Q(z)$ is starlike univalent in $\mathbb{U}$,*

(2) $\Re\left(\frac{zh'(z)}{Q(z)}\right) > 0$ *for $z \in \mathbb{U}$.*

If ξ is analytic in $\mathbb{U}$, with $\xi(0) = q(0)$, $\xi(\mathbb{U}) \subset D$ and

$$\theta(\xi(z)) + z\xi'(z)\phi(\xi(z)) \prec \theta(q(z)) + zq'(z)\phi(q(z)), \tag{5}$$

then $\xi \prec q$ and q is the best dominant of (5)*.*

Lemma 2 ([9])**.** *Let q be a convex univalent function in $\mathbb{U}$and let $\mu \in \mathbb{C}$, $\nu \in \mathbb{C}\backslash\{0\}$ with*

$$\Re\left(1 + \frac{zq''(z)}{q'(z)}\right) > \max\left\{0, -Re\left(\frac{\mu}{\nu}\right)\right\}.$$

If ξ is analytic in $\mathbb{U}$ and

$$\mu\xi(z)+\nu z\xi'(z)\prec\mu q(z)+\nu zq'(z), \tag{6}$$

then $\xi\prec q$ and q is the best dominant of (6).

Lemma 3 ([9]). *Let q be convex univalent in $\mathbb{U}$ and let $\nu\in\mathbb{C}$. Further assume that $\Re(\nu)>0$. If $\xi\in H[q(0),1]\cap Q$ and $\xi(z)+\nu z\xi'(z)$ is univalent in $\mathbb{U}$, then*

$$q(z)+\nu zq'(z)\prec\xi(z)+\nu z\xi'(z), \tag{7}$$

which implies that $q\prec\xi$ and q is the best subordinant of (7).

Lemma 4 ([5]). *Let q be convex univalent in the unit disk $\mathbb{U}$ and let θ and ϕ be analytic in a domain D containing $q(\mathbb{U})$. Suppose that*

(1) $\Re\left(\frac{\theta'(q(z))}{\phi(q(z))}\right)>0$ *for* $z\in\mathbb{U}$,

(2) $Q(z)=zq'(z\phi(q(z)))$ *is starlike univalent in* $\mathbb{U}$.

If $\xi\in H[q(0),1]\cap Q$, with $\xi(\mathbb{U})\subset D$, $\phi(\xi(z))+z\xi'(z)\phi(\xi(z))$ is univalent in $\mathbb{U}$ and

$$\theta(q(z))+zq'(z)\phi(q(z))\prec\theta(\xi(z))+z\xi'(z)\phi(\xi(z)), \tag{8}$$

then $q\prec\xi$ and q is the best subordinant of (8).

2. MAIN RESULTS

Theorem 1. *Let q be convex univalent in $\mathbb{U}$ with $q(0)=1$, $\sigma\in\mathbb{C}\backslash\{0\}$, $\gamma>0$ and suppose that q satisfies*

$$\Re\left(1+\frac{zq''(z)}{q'(z)}\right)>\max\left\{0,-\Re\left(\frac{\gamma}{\sigma}\right)\right\}. \tag{9}$$

If $f\in\mathcal{A}$ satisfies the subordination

$$\begin{aligned}&\left[1-\sum_{m=1}^{k}\binom{k}{m}(-1)^{m+1}\left(\left(\frac{\alpha}{\beta}\right)^{m}+1\right)\right]\left(\frac{W_{\alpha,\beta}^{k,\delta}f(z)}{z}\right)^{\gamma}\\&+\sum_{m=1}^{k}\binom{k}{m}(-1)^{m+1}\left(\left(\frac{\alpha}{\beta}\right)^{m}+1\right)\left(\frac{W_{\alpha,\beta}^{k,\delta}f(z)}{z}\right)^{\gamma}\left(\frac{W_{\alpha,\beta}^{k,\delta+1}f(z)}{W_{\alpha,\beta}^{k,\delta}f(z)}\right)\\&\prec q(z)+\frac{\sigma}{\gamma}zq'(z),\end{aligned} \tag{10}$$

then

$$\left(\frac{W_{\alpha,\beta}^{k,\delta}f(z)}{z}\right)^{\gamma}\prec q(z) \tag{11}$$

and q is the best dominant of (10).

Proof. Define the function ξ by

$$\xi(z)=\left(\frac{W_{\alpha,\beta}^{k,\delta}f(z)}{z}\right)^{\gamma},\quad (z\in\mathbb{U}). \tag{12}$$

Differentiating (12) logarithmically with respect to z, we get

$$\frac{z\xi'(z)}{\xi(z)}=\gamma\left(\frac{z\left(W_{\alpha,\beta}^{k,\delta}f(z)\right)'}{W_{\alpha,\beta}^{k,\delta}f(z)}-1\right).$$

Now, in view of (4), we obtain the following subordination

$$\frac{z\xi'(z)}{\xi(z)}=\sum_{m=1}^{k}\binom{k}{m}(-1)^{m+1}\left(\left(\frac{\alpha}{\beta}\right)^{m}+1\right)\left(\frac{W_{\alpha,\beta}^{k,\delta+1}f(z)}{W_{\alpha,\beta}^{k,\delta}f(z)}-1\right).$$

Therefore,

$$\frac{z\xi'(z)}{\gamma}=\sum_{m=1}^{k}\binom{k}{m}(-1)^{m+1}\left(\left(\frac{\alpha}{\beta}\right)^{m}+1\right)\times$$
$$\times\left(\frac{W_{\alpha,\beta}^{k,\delta}f(z)}{z}\right)^{\gamma}\left(\frac{W_{\alpha,\beta}^{k,\delta+1}f(z)}{W_{\alpha,\beta}^{k,\delta}f(z)}-1\right).$$

The subordination (10) from the hypothesis becomes

$$\xi(z)+\frac{\sigma}{\gamma}z\xi'(z)\prec q(z)+\frac{\sigma}{\gamma}zq'(z).$$

Hence, an application of Lemma 2 with $\mu=1$ and $\nu=\frac{\sigma}{\gamma}$, we obtain (11). $\square$

Theorem 2. *Let $\eta,\tau\in\mathbb{C}$, $\gamma>0$, $\lambda\in\mathbb{C}\backslash\{0\}$ and q be convex univalent in $\mathbb{U}$ with $q(0)=1$, $q(z)\neq 0$ $(z\in\mathbb{U})$ and assume that q satisfies*

$$\Re\left(1+\frac{\tau}{\lambda}q(z)+\frac{zq''(z)}{q'(z)}-\frac{zq'(z)}{q(z)}\right)>0. \tag{13}$$

Suppose that $\frac{zq'(z)}{q(z)}$ is starlike univalent in $\mathbb{U}$. If $f\in\mathcal{A}$ satisfies

$$\Omega(\eta,\tau,\gamma,\lambda,k,\delta,\alpha,\beta;z)\prec\eta+\tau q(z)+\lambda\frac{zq'(z)}{q(z)}, \tag{14}$$

where

(15)

$$\Omega(\eta,\tau,\gamma,\lambda,k,\delta,\alpha,\beta;z)=\eta+\tau\left(\frac{W_{\alpha,\beta}^{k,\delta+1}f(z)}{W_{\alpha,\beta}^{k,\delta}f(z)}\right)^{\gamma}$$
$$+\gamma\lambda\sum_{m=1}^{k}\binom{k}{m}(-1)^{m+1}\left(\left(\frac{\alpha}{\beta}\right)^{m}+1\right)\left(\frac{W_{\alpha,\beta}^{k,\delta+2}f(z)}{W_{\alpha,\beta}^{k,\delta+1}f(z)}-\frac{W_{\alpha,\beta}^{k,\delta+1}f(z)}{W_{\alpha,\beta}^{k,\delta}f(z)}\right),$$

then

$$\left(\frac{W_{\alpha,\beta}^{k,\delta+1} f(z)}{W_{\alpha,\beta}^{k,\delta} f(z)}\right)^{\gamma} \prec q(z)$$

and q is the best dominant of (14).

Proof. Define the function ξ by

$$\xi(z) = \left(\frac{W_{\alpha,\beta}^{k,\delta+1} f(z)}{W_{\alpha,\beta}^{k,\delta} f(z)}\right)^{\gamma}, \quad (z \in \mathbb{U}). \tag{16}$$

By a straightforward computation and using (4), we have

$$\eta + \tau\xi(z) + \lambda\frac{z\xi'(z)}{\xi(z)} = \Omega(\eta, \tau, \gamma, \lambda, k, \delta, \alpha, \beta; z), \tag{17}$$

where $\Omega(\eta, \tau, \gamma, \lambda, k, \delta, \alpha, \beta; z)$ is given by (15). From (14) and (17), we obtain

$$\eta + \tau\xi(z) + \lambda\frac{z\xi'(z)}{\xi(z)} \prec \eta + \tau q(z) + \lambda\frac{zq'(z)}{q(z)}.$$

By setting

$$\theta(w) = \eta + \tau w \text{ and } \phi(w) = \frac{\lambda}{w},\ w \neq 0,$$

we see that $\theta(w)$ is analytic in $\mathbb{C}$, $\phi(w)$ is analytic in $\mathbb{C}\backslash\{0\}$ and that $\phi(w) \neq 0$, $w \in \mathbb{C}\backslash\{0\}$. Also, we get

$$Q(z) = zq'(z)\phi(q(z)) = \lambda\frac{zq'(z)}{q(z)}$$

and

$$h(z) = \theta(q(z)) + Q(z) = \eta + \tau q(z) + \lambda\frac{zq'(z)}{q(z)}.$$

It is clear that $Q(z)$ is starlike univalent in $\mathbb{U}$,

$$\Re\left(\frac{zh'(z)}{Q(z)}\right) = \Re\left(1 + \frac{\tau}{\lambda}q(z) + \frac{zq''(z)}{q'(z)} - \frac{zq'(z)}{q(z)}\right) > 0.$$

Thus, by Lemma 1, we get $\xi(z) \prec q(z)$. By using (16), we obtain the desired result. □

Theorem 3. *Let q be convex univalent in $\mathbb{U}$ with $q(0) = 1$, $\gamma > 0$ and $\Re(\sigma) > 0$. Let $f \in \mathcal{A}$ satisfies*

$$\left(\frac{W_{\alpha,\beta}^{k,\delta} f(z)}{z}\right)^{\gamma} \in H[q(0), 1] \cap Q$$

and

$$\left[1 - \sigma\sum_{m=1}^{k}\binom{k}{m}(-1)^{m+1}\left(\left(\frac{\alpha}{\beta}\right)^{m} + 1\right)\right]\left(\frac{W_{\alpha,\beta}^{k,\delta} f(z)}{z}\right)^{\gamma}$$

$$+\sigma\sum_{m=1}^{k}\binom{k}{m}(-1)^{m+1}\left(\left(\frac{\alpha}{\beta}\right)^{m}+1\right)\left(\frac{W_{\alpha,\beta}^{k,\delta}f(z)}{z}\right)^{\gamma}\left(\frac{W_{\alpha,\beta}^{k,\delta+1}f(z)}{W_{\alpha,\beta}^{k,\delta}f(z)}\right)$$

be univalent in $\mathbb{U}$. *If*

$$\begin{aligned}
&q(z)+\frac{\sigma}{\gamma}zq'(z)\\
&\prec\left[1-\sigma\sum_{m=1}^{k}\binom{k}{m}(-1)^{m+1}\left(\left(\frac{\alpha}{\beta}\right)^{m}+1\right)\right]\left(\frac{W_{\alpha,\beta}^{k,\delta}f(z)}{z}\right)^{\gamma}\\
&+\sigma\sum_{m=1}^{k}\binom{k}{m}(-1)^{m+1}\left(\left(\frac{\alpha}{\beta}\right)^{m}+1\right)\left(\frac{W_{\alpha,\beta}^{k,\delta}f(z)}{z}\right)^{\gamma}\times\\
&\times\left(\frac{W_{\alpha,\beta}^{k,\delta+1}f(z)}{W_{\alpha,\beta}^{k,\delta}f(z)}\right),
\end{aligned}\tag{18}$$

then

$$q(z)\prec\left(\frac{W_{\alpha,\beta}^{k,\delta}f(z)}{z}\right)^{\gamma}\tag{19}$$

and q *is the best subordinant of (18).*

Proof. Let ξ be defined by (12), then differentiating ξ with respect to z, we get

$$\frac{z\xi'(z)}{\xi(z)}=\gamma\left(\frac{z\left(W_{\alpha,\beta}^{k,\delta}f(z)\right)'}{W_{\alpha,\beta}^{k,\delta}f(z)}-1\right).\tag{20}$$

By using (4) for $\left(W_{\alpha,\beta}^{k,\delta}f(z)\right)'$, in (20), we have

$$\begin{aligned}
&\left[1-\sigma\sum_{m=1}^{k}\binom{k}{m}(-1)^{m+1}\left(\left(\frac{\alpha}{\beta}\right)^{m}+1\right)\right]\left(\frac{W_{\alpha,\beta}^{k,\delta}f(z)}{z}\right)^{\gamma}\\
&+\sigma\sum_{m=1}^{k}\binom{k}{m}(-1)^{m+1}\left(\left(\frac{\alpha}{\beta}\right)^{m}+1\right)\left(\frac{W_{\alpha,\beta}^{k,\delta}f(z)}{z}\right)^{\gamma}\times\\
&\times\left(\frac{W_{\alpha,\beta}^{k,\delta+1}f(z)}{W_{\alpha,\beta}^{k,\delta}f(z)}\right)=\xi(z)+\frac{\sigma}{p\gamma}z\xi'(z).
\end{aligned}\tag{21}$$

From (18) and (21), we get

$$q(z)+\frac{\sigma}{\gamma}zq'(z)\prec\xi(z)+\frac{\sigma}{\gamma}z\xi'(z).$$

Hence, by using Lemma 3 with $\mu=1$ and $\nu=\frac{\sigma}{\gamma}$, we obtain (19). □

Theorem 4. *Let $\eta \in \mathbb{C}$, $\gamma > 0$, $\lambda \in \mathbb{C}\backslash\{0\}$ and q be convex univalent in $\mathbb{U}$ with $q(0) = 1$, $q(z) \neq 0$ $(z \in \mathbb{U})$ and assume that q satisfies*

$$\Re\left(\frac{\tau}{\lambda} q(z)\right) > 0. \tag{22}$$

Suppose that $\frac{zq'(z)}{q(z)}$ is starlike univalent in $\mathbb{U}$. If $f \in \mathcal{A}$ satisfies

$$\left(\frac{W_{\alpha,\beta}^{k,\delta+1} f(z)}{W_{\alpha,\beta}^{k,\delta} f(z)}\right)^{\gamma} \in H[q(0), 1] \cap Q$$

and $\Omega(\eta, \tau, \gamma, \lambda, k, \delta, \alpha, \beta; z)$ is univalent in $\mathbb{U}$, where $\Omega(\eta, \tau, \gamma, \lambda, k, \delta, \alpha, \beta; z)$ is given by (15). If

$$\eta + \tau q(z) + \lambda \frac{zq'(z)}{q(z)} \prec \Omega(\eta, \tau, \gamma, \lambda, k, \delta, \alpha, \beta; z), \tag{23}$$

then

$$q(z) \prec \left(\frac{W_{\alpha,\beta}^{k,\delta+1} f(z)}{W_{\alpha,\beta}^{k,\delta} f(z)}\right)^{\gamma}$$

and q is the best subordinant of (23).

Proof. Assume that the function ξ be defined by (16). By a straightforward computation, we have

$$\Omega(\eta, \tau, \gamma, \lambda, k, \delta, \alpha, \beta; z) = \eta + \tau\xi(z) + \lambda \frac{z\xi'(z)}{\xi(z)}, \tag{24}$$

where $\Omega(\eta, \tau, \gamma, \lambda, k, \delta, \alpha, \beta; z)$ is given by (15). From (23) and (24), we obtain

$$\eta + \tau q(z) + \lambda \frac{zq'(z)}{q(z)} \prec \eta + \tau\xi(z) + \lambda \frac{z\xi'(z)}{\xi(z)}.$$

By setting $\theta(w) = \eta + \tau w$ and $\phi(w) = \frac{\lambda}{w}$, $w \neq 0$, we see that $\theta(w)$ is analytic in $\mathbb{C}$, $\phi(w)$ is analytic in $\mathbb{C}\backslash\{0\}$ and that $\phi(w) \neq 0$, $w \in \mathbb{C}\backslash\{0\}$. Also, we get

$$Q(z) = zq'(z)\,\phi(q(z)) = \lambda \frac{zq'(z)}{q(z)}.$$

It is clear that $Q(z)$ is starlike univalent in $\mathbb{U}$,

$$\Re\left(\frac{\theta'(q(z))}{\phi(q(z))}\right) = \Re\left(\frac{\tau}{\lambda} q(z)\right) > 0.$$

Thus, by Lemma 4, we get $q(z) \prec \xi(z)$. By using (16), we obtain the desired result. $\square$

Concluding the results of differential subordination and superordination, we state the following "sandwich results".

Theorem 5. *Let q_1 and q_2 be convex univalent in $\mathbb{U}$ with $q_1(0) = q_2(0) = 1$. Suppose q_2 satisfies* (9), $\gamma > 0$ *and* $\Re(\sigma) > 0$. *Let $f \in \mathcal{A}$ satisfies*

$$\left(\frac{W_{\alpha,\beta}^{k,\delta} f(z)}{z}\right)^{\gamma} \in H[1,1] \cap Q$$

and

$$\left[1 - \sigma \sum_{m=1}^{k} \binom{k}{m} (-1)^{m+1} \left(\left(\frac{\alpha}{\beta}\right)^m + 1\right)\right] \left(\frac{W_{\alpha,\beta}^{k,\delta} f(z)}{z}\right)^{\gamma}$$
$$+ \sigma \sum_{m=1}^{k} \binom{k}{m} (-1)^{m+1} \left(\left(\frac{\alpha}{\beta}\right)^m + 1\right) \left(\frac{W_{\alpha,\beta}^{k,\delta} f(z)}{z}\right)^{\gamma} \left(\frac{W_{\alpha,\beta}^{k,\delta+1} f(z)}{W_{\alpha,\beta}^{k,\delta} f(z)}\right)$$

be univalent in $\mathbb{U}$. If

$$q_1(z) + \frac{\sigma}{\gamma} z q_1'(z)$$
$$\prec \left[1 - \sigma \sum_{m=1}^{k} \binom{k}{m} (-1)^{m+1} \left(\left(\frac{\alpha}{\beta}\right)^m + 1\right)\right] \left(\frac{W_{\alpha,\beta}^{k,\delta} f(z)}{z}\right)^{\gamma}$$
$$+ \sigma \sum_{m=1}^{k} \binom{k}{m} (-1)^{m+1} \left(\left(\frac{\alpha}{\beta}\right)^m + 1\right) \left(\frac{W_{\alpha,\beta}^{k,\delta} f(z)}{z}\right)^{\gamma} \left(\frac{W_{\alpha,\beta}^{k,\delta+1} f(z)}{W_{\alpha,\beta}^{k,\delta} f(z)}\right)$$
$$\prec q_2(z) + \frac{\sigma}{\gamma} z q_2'(z),$$

then

$$q_1(z) \prec \left(\frac{W_{\alpha,\beta}^{k,\delta} f(z)}{z}\right)^{\gamma} \prec q_2(z)$$

and q_1 and q_2 are, respectively, the best subordinant and the best dominant.

Theorem 6. *Let q_1 and q_2 be convex univalent in $\mathbb{U}$ with $q_1(0) = q_2(0) = 1$. Suppose q_1 satisfies (22) and q_2 satisfies (13). Let $f \in \mathcal{A}$ satisfies*

$$\left(\frac{W_{\alpha,\beta}^{k,\delta+1} f(z)}{W_{\alpha,\beta}^{k,\delta} f(z)}\right)^{\gamma} \in H[1,1] \cap Q$$

and $\Omega(\eta, \tau, \gamma, \lambda, k, \delta, \alpha, \beta; z)$ is univalent in $\mathbb{U}$, where $\Omega(\eta, \tau, \gamma, \lambda, k, \delta, \alpha, \beta; z)$ is given by (15). If

$$\eta + \tau q_1(z) + \lambda \frac{z q_1'(z)}{q_1(z)} \prec \Omega(\eta, \tau, \gamma, \lambda, k, \delta, \alpha, \beta; z)$$
$$\prec \eta + \tau q_2(z) + \lambda \frac{z q_2'(z)}{q_2(z)},$$

then

$$q_1(z) \prec \left(\frac{W_{\alpha,\beta}^{k,\delta+1} f(z)}{W_{\alpha,\beta}^{k,\delta} f(z)} \right)^{\gamma} \prec q_2(z)$$

and q_1 and q_2 are, respectively, the best subordinant and the best dominant.

Remark 2. By selecting the particular values of δ, k, α and β, we can derive a number of known results. Some of them are given below:

(1) Taking $\delta = 0$ in Theorem 1, we obtain the results obtained by Murugusundaramoorthy and Magesh [10, Corollary 3.3],

(2) Putting $k = 1$, $\alpha = 1 - \beta$ and $\beta \geq 0$ in Theorems 1, 3 and 5, we get the results obtaioned by Răducanu and Nechita [11, Theorem 3.1, Theorem 3.6, Theorem 3.9],

(3) Setting $\alpha = 0$ and $k = \beta = 1$ in Theorems 1, 3 and 5, we get the results obtained by Răducanu and Nechita [11, Corollary 3.3, Corollary 3.8, Corollary 3.11].

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